

How many solutions?

Hung-yi Lee

Reference

- Textbook: Chapter 1.7

Review

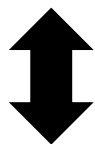
Given a system of linear equations with m equations and n variables

$$Ax = b \quad A: m \times n \quad x \in R^n \quad b \in R^m$$

Is b a linear combination of columns of A ?

Is b in the span of the columns of A ?

NO



No
solution

YES



Have
solution

How many solutions?

Today

Given a system of linear equations with
 m equations and n variables

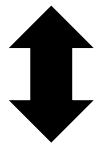
$$Ax = b \quad A: m \times n \quad x \in R^n \quad b \in R^m$$

Is b a linear combination
of columns of A ?

Is b in the span of the
columns of A ?

NO

YES



No
solution

Other cases?

The columns of A
are *independent*.

Rank $A = n$

Nullity $A = 0$

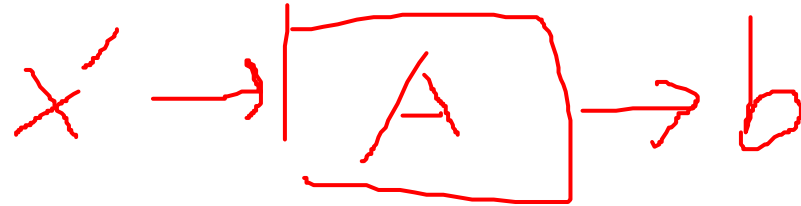
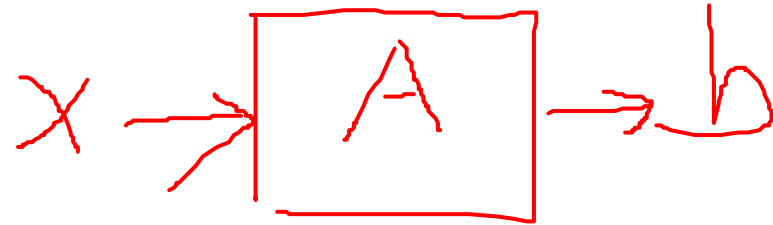
Unique solution

The columns of A
are *dependent*.

Rank $A < n$

Nullity $A > 0$

Infinite solution



(依賴的、不獨立的)

Dependent and Independent

(獨立的、自主的)

Definition

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear dependent

Find one



Obtain many

- If there exist scalars x_1, x_2, \dots, x_n , **not all zero**, such that

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

- A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is linear independent

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n = \mathbf{0}$$

$$\text{Only if } x_1 = x_2 = \dots = x_n = 0$$

unique

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_i that is a linear combination of other vectors

$$\left\{ \begin{bmatrix} -4 \\ 12 \\ 6 \end{bmatrix}, \begin{bmatrix} -10 \\ 30 \\ 15 \end{bmatrix} \right\}$$

Dependent or Independent?

$$\left\{ \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} \right\}$$

Dependent or Independent?

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_i that is a linear combination of other vectors

$$\left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\} \quad \text{Dependent or Independent?}$$

Zero vector is the linear combination of any other vectors

Any set contains zero vector would be linear dependent

How about a set with only one vector?

Dependent and Independent

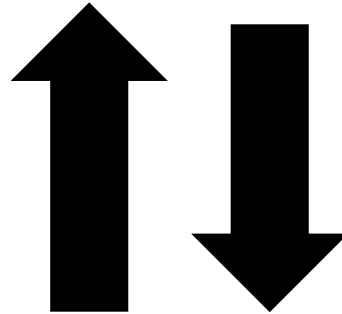
Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, if there exists any \mathbf{a}_j that is a linear combination of other vectors

$$2\mathbf{a}_i + \mathbf{a}_j + 3\mathbf{a}_k = \mathbf{0}$$

$$2\mathbf{a}_i + \mathbf{a}_j = -3\mathbf{a}_k$$

$$\left(-\frac{2}{3}\right)\mathbf{a}_i + \left(-\frac{1}{3}\right)\mathbf{a}_j = \mathbf{a}_k$$



$$\mathbf{a}_{i'} = 3\mathbf{a}_{j'} + 4\mathbf{a}_{k'}$$

$$\mathbf{a}_{i'} - 3\mathbf{a}_{j'} - 4\mathbf{a}_{k'} = \mathbf{0}$$

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$, there exists scalars x_1, x_2, \dots, x_n , that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$.

Intuition

Dependent:

Once we have solution, we have infinite.

- Intuitive link between dependence and the number of solutions

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \quad 1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite
Solution

Proof

- Columns of A are **dependent** \rightarrow If $Ax=b$ have solution, it will have Infinite Solutions

- If $Ax=b$ have Infinite solutions \rightarrow Columns of A are dependent

Proof

Homogeneous linear equations

$$Ax = \mathbf{0} \quad A = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(always having $\mathbf{0}$ as solution)

Based on the definition

A set of n vectors $\{a_1, a_2, \dots, a_n\}$ is linear dependent



$Ax = \mathbf{0}$ have non-zero solution

infinite

A set of n vectors $\{a_1, a_2, \dots, a_n\}$ is linear independent



$Ax = \mathbf{0}$ only have zero solution

Proof

- Columns of A are **dependent** \rightarrow If $Ax=b$ have solution, it will have Infinite solutions

We can find non-zero solution u such that $Au = \mathbf{0}$

There exists v such that $Av = b$

$$A(u + v) = b$$

$u + v$ is another solution different to v

- If $Ax=b$ have Infinite solutions \rightarrow Columns of A are dependent

$$u \neq v \quad \left. \begin{array}{l} Au = b \\ Av = b \end{array} \right\}$$

$$\underline{A(u - v)} = \mathbf{0}$$

Non-zero

Rank and Nullity

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} -3 & 2 & -1 \\ 7 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$$

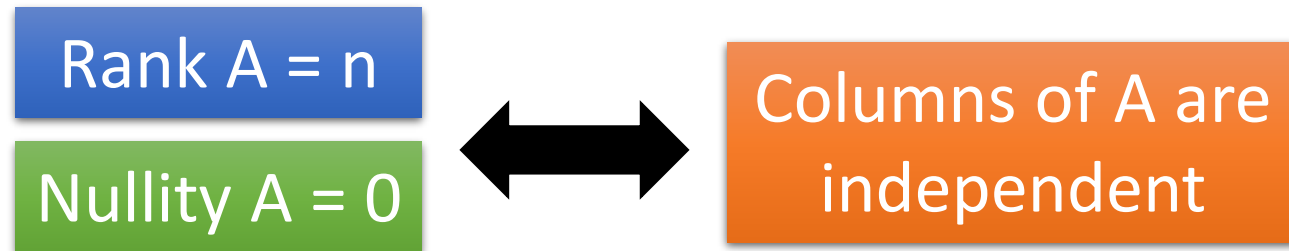
$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$[6]$$

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

If A is a $m \times n$ matrix:



Conclusion

$$A\mathbf{x} = \mathbf{b}$$

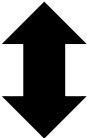
$$A: m \times n \quad \mathbf{x} \in \mathbb{R}^n \quad \mathbf{b} \in \mathbb{R}^m$$

Is \mathbf{b} a linear combination of columns of A ?

Is \mathbf{b} in the span of the columns of A ?

NO

YES


No
solution

The columns of A
are *independent*.

$$\text{Rank } A = n$$

$$\text{Nullity } A = 0$$

Unique solution

The columns of A
are *dependent*.

$$\text{Rank } A < n$$

$$\text{Nullity } A > 0$$

Infinite solution

Conclusion

The columns of A
are *independent*.

Rank $A = n$

Nullity $A = 0$

$$A: m \times n$$

$$x \in R^n \quad b \in R^m$$

NO

YES

Is b a linear combination
of columns of A ?

Is b in the span of the
columns of A ?

Is b a linear combination
of columns of A ?

Is b in the span of the
columns of A ?

NO

YES

No
solution

Infinite
solution

NO

YES

No
solution

Unique
solution

Question

- True or False

- If the columns of A are linear independent, then $Ax=b$ has unique solution.
- If the columns of A are linear independent, then $Ax=b$ has at most one solution.
- If the columns of A are linear dependent, then $Ax=b$ has infinite solution.
- If the columns of A are linear independent, then $Ax=0$ (homogeneous equation) has unique solution.
- If the columns of A are linear dependent, then $Ax=0$ (homogeneous equation) has infinite solution.